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Static and dynamic beam forms of the loss of stability of a long orthotropic cylindrical shell under external pressure☆

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ABSTRACT

A cylindrical shell with end sections which are closed and supported by hinges, in accordance with the concepts of the rod theory, is considered to be under the action of an omnidirectional external pressure which remains normal to the lateral surface during the deformation process. It is shown that, for such shells, the previously constructed consistent equations of the momentless theory, reduced using the Timoshenko shear model to the one-dimensional equations of the rod theory, describe three forms of loss of stability: (1) static loss of stability, which occures through a bending mode from the action of the total end axial compression force since, under the clamping conditions considered, its non-conservative part cannot perform work on deflections of the axial line; (2) also a static loss of stability but one which occurs through a purely shear mode with the conversion of a cylinder with normal sections into a cylinder with parallel sloping sections and a corresponding critical load which is independent of the length of the shell; (3) dynamic loss of stability which occurs through a bending-shear form and can only be revealed by a dynamic method using an improved shear model.

The generalizing results of investigations into the stability of cylindrical shells, carried out by many authors, have been presented, in particular, by Grigolyuk and Kabanov.[1] However, the majority of them only refer to investigations of the classical bending forms of loss of stability (FLS). It has been established[2–4] that, together with these FLS, a number of other (non-classical) forms also exist for cylindrical shells which can be realized earlier than the classical bending forms for the same or other forms of loading and certain combinations of the governing physico-mechanical and geometrical parameters of the shell.

As an extension of the preceding investigations,[2–4] below we consider the case when an omnidirectional external pressure, which remains normal to the lateral surface during deformation, acts on a cylindrical shell with closed end sections which are supported by hinges.

1. The action of a constant-direction external pressure on a shell

Consider a cylindrical shell of length L, thickness t and radius of the middle surface R, which is acted upon by an external pressure p. We introduce the coordinate x, measured along the generatrix of the shell, and the polar angle θ, measured from a certain plane. The shell material is assumed to be orthotropic with elasticity characteristics E_1, E_2, G_{12}, $\nu_{12} = \nu_{21} E_1/E_2$. It is well known that, in the momentless approximation, shear compressive stresses of the form

$$T_{11}^0 = -pR/2, \quad T_{22}^0 = -pR \tag{1.1}$$

occur in the shell if the pressure p acts both on the lateral and end surface of the shell, or of the form

$$T_{11}^0 = 0, \quad T_{22}^0 = -pR \tag{1.2}$$

if only the lateral surface is loaded.

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When the direction of the action of the pressure remains fixed (a “dead”[5] force) during the deformation process, the following variational equation holds

$$\delta\Pi = \int_0^L\int_0^{2\pi}\sum_{i,j=1}^{2}(S_{ij}\delta e_{ij} + S_{i3}\delta\omega_i)Rdxd\theta = 0 \tag{1.3}$$

for investigating certain non-classical forms of loss of stability (FLS) of the shell considered with the limits of the use of the relations of momentless theory, constructed in the consistent quadratic approximation.[2]

$$S_{11} = B_{11}(e_{11} + \nu_{21}e_{22}),\quad S_{12} = (B_{12} + T_{11}^0)e_{12} + B_{12}e_{21},\quad S_{13} = T_{11}^0\omega_1;\quad \overrightarrow{\underleftarrow{1, 2}}$$
$$B_{11} = E_1t/(1-\nu_{12}\nu_{21}),\quad \overrightarrow{\underleftarrow{1, 2}};\quad B_{12} = B_{21} = G_{12}t \tag{1.4}$$

The quantities e_{ij}, ω_i are related to the displacements *u*, *v* and *w* of the points of the middle surface by the following equations

$$e_{11} = u_{,x},\quad e_{12} = \upsilon_{,x},\quad \omega_1 = w_{,x},\quad e_{21} = \frac{u_{,\theta}}{R},\quad e_{22} = \frac{\upsilon_{,\theta} + w}{R},\quad \omega_2 = \frac{w_{,\theta} - \upsilon}{R} \tag{1.5}$$

The subscripts after a comma denote partial derivatives with respect to the corresponding independent variables *x* and θ.

We introduce the notation

$$\tilde{T}_{ij} = T_{ij}/R,\quad \tilde{B}_{ij} = B_{ij}/R$$

Then, starting from Eq. (1.3) and using relations (1.4) and (1.5), the system of homogeneous differential equations for the static neutral equilibrium of the shell

$$f_1 = RS_{11,x} + S_{21,\theta} = L_1(u, \upsilon, w) + \tilde{T}_{22}^0u_{,\theta\theta} = 0$$
$$f_2 = RS_{12,x} + S_{22,\theta} + S_{13} = L_2(u, \upsilon, w) + RT_{11}^0\upsilon_{,xx} + \tilde{T}_{22}^0(w_{,\theta} - \upsilon) = 0$$
$$f_3 = RS_{13,x} + S_{23,\theta} - S_{22} = L_3(u, \upsilon, w) + RT_{11}^0w_{,xx} + \tilde{T}_{22}^0(w_{,\theta\theta} - \upsilon_{,\theta}) = 0 \tag{1.6}$$

is established, where

$$L_1(u, \upsilon, w) = B_{11}Ru_{,xx} + (\nu_{21}B_{11} + B_{12})\upsilon_{,x\theta} + \tilde{B}_{12}u_{,\theta\theta} + \nu_{21}B_{11}w_{,x}$$
$$L_2(u, \upsilon, w) = B_{12}R\upsilon_{,xx} + (\nu_{12}B_{22} + B_{12})u_{,x\theta} + \tilde{B}_{22}(\upsilon_{,\theta\theta} + w_{,\theta})$$
$$L_3(u, \upsilon, w) = -\nu_{12}B_{22}u_{,x} - \tilde{B}_{12}(\upsilon_{,\theta} + w) \tag{1.7}$$

When the initial conditions in the shell, due to action of an external pressure in a fixed direction, are defined by equalities (1.2), the solutions of the problem of the possible non-classical FLS have been constructed earlier.[2] One of them corresponds to the realization of a FLS in the shell with displacements with zero variability in the peripheral direction, that is, when $u = u(x)$, $\upsilon = \upsilon(x)$, $w = w(x)$. In this case, for the forces defined by equalities (1.1), Eq. (1.6) take the form

$$f_1 = u_{,xx} + \frac{\nu_{21}}{R}w_{,x} = 0$$
$$f_2 = R^2\left(\tilde{B}_{12} - \frac{p}{2}\right)\upsilon_{,xx} + p\upsilon = 0$$
$$f_3 = \nu_{12}B_{22}u_{,x} + \tilde{B}_{22}w + \frac{pR^2}{2}w_{,xx} = 0 \tag{1.8}$$

By analogy with the results previously obtained,[2] the first bifurcation value

$$p_{(1)}^* = \frac{B_{12}R\pi^2}{L^2(1 + \lambda^2/2)};\quad \lambda = \frac{R\pi}{L} \tag{1.9}$$

follows from the second equation of (1.8) for the pressure. This value corresponds to a shear FLS of the shell with displacements of the form

$$\upsilon = V_0\sin(\pi x/R),\quad u = 0,\quad w = 0 \tag{1.10}$$

and the second bifurcation value

$$p_{(2)}^* = \frac{2B_2}{n^2\lambda^2},\quad n = 1, 2, \ldots;\quad B_2 = E_2t \tag{1.11}$$

which is not present in the case previously considered,[2] follows from the system consisting of the first and third equations of (1.8). Moreover, it is obvious that, when $n \to \infty$, we have $p^*_{(2)} \to 0$. If $\lambda = 0$ is taken as equality (1.9), we arrive at the formulae obtained earlier.[2] Consequently, the action of the external pressure on the end sections $x = 0$ and $x = L$, which, as can be seen, reduces the value of the critical pressure, is taken account of by the term $\lambda^2/2$ in the denominator.

The second form of solution corresponds to zero variability of the functions u, v and w in the axial direction, for which Eq. (1.6) take the form

$$f_1 = (\tilde{B}_{12} - p)u_{,\theta\theta} = 0$$
$$f_2 = \tilde{B}_{22}(v_{,\theta\theta} + w_{,\theta}) - p(w_{,\theta} - v) = 0$$
$$f_3 = \tilde{B}_{22}(v_{,\theta} + w) + p(w_{,\theta\theta} - v_{,\theta}) = 0 \quad (1.12)$$

As before,[2] the bifurcation value

$$p^*_{(3)} = p^*_c = \tilde{B}_{12} \quad (1.13)$$

follows from the first equation of (1.12). This value corresponds to a purely shear FLS when, due to the displacements

$$u = U_0\cos\theta, \quad v = 0, \quad w = 0$$

the initial cylinder with normal sections is transformed, after static loss of stability, into a cylinder with sloping sections but the system, consisting of the second and third equations of (1.12), is of absolutely no interest within the limits of the assumption made regarding the invariability of the displacement function.

As before,[2] we take the representations

$$u = U(x)\cos\theta, \quad v = V(x)\sin\theta, \quad w = W(x)\cos\theta \quad (1.14)$$

for the displacements and, using these, Eqs. (1.6) take the form

$$B_{11}RU'' + (\nu_{21}B_{11} + B_{12})V' - (\tilde{B}_{12} - p)U' + \nu_{21}B_{11}W' = 0$$
$$R^2\left(B_{12} - \underline{\frac{p}{2}}\right)V'' - (\nu_{12}B_{22} + B_{12})U' - (\tilde{B}_{22} - p)(V + W) = 0$$
$$-\nu_{12}B_{22}U' - (\tilde{B}_{22} - p)(V + W) - \underline{\frac{p}{2}W''} = 0 \quad (1.15)$$

If the underlined terms are discarded in system of equations (1.15), which is equivalent to determining the initial stresses using formulae (1.2), then the bifurcation value

$$p^*_{(4)} = \frac{\pi^2 B_1 R}{(1 + \nu_{12})L^2}; \quad B_1 = E_1 t \quad (1.16)$$

is established for the boundary conditions $U'(0) = U'(L) = 0$, which corresponds to the solution

$$U = U_0\sin(\pi x/L), \quad V \neq 0, \quad W \neq 0$$

This FLS is completely analogous to the bending FLS of a rod-strip under uniform transverse compression[6] and is realized without deformations of the transverse displacements.

For the case being considered, when the boundary conditions are formulated in the form

$$S_{11}(0) = S_{11}(L) = 0, \quad v(0) = v(L) = w(0) = w(L) = 0$$

we take the representations

$$u = U_n\cos\lambda_n x, \quad v = V_n\sin\lambda_n x, \quad w = W_n\sin\lambda_n x; \quad \lambda_n = n\pi/L, \quad n = 1, 2, \ldots \quad (1.17)$$

for the displacements and, substituting these into Eq. (1.15), we obtain the homogeneous algebraic system

$$a_{i1}U_n + a_{i2}V_n + a_{i3}W_n = 0, \quad i = 1, 2, 3 \quad (1.18)$$

where

$$a_{11} = -B_{11}R\lambda_n^2 - \tilde{B}_{12} + p, \quad a_{12} = (\nu_{21}B_{11} + B_{12})\lambda_n, \quad a_{13} = \nu_{21}B_{11}\lambda_n$$
$$a_{22} = -R^2(\tilde{B}_{12} - p/2)\lambda_n^2 - \tilde{B}_{22} + p, \quad a_{23} = \tilde{B}_{22} + p, \quad a_{33} = p\lambda_n^2/2 - \tilde{B}_{22} + p \quad (1.19)$$

It is not possible to determine the bifurcation value of p^*, corresponding to a beam bending FLS in the form of a simple analytical formula from the system of Eq. (1.18). Hence, in order to simplify the problem without any loss in its content in a transverse section of the shell

x = const, we will represent the displacement vector $\mathbf{u}$ of an arbitrary point on the middle surface in the form[3,4]

$$\begin{aligned}\mathbf{u} &= \mathbf{U}(x) + \boldsymbol{\varphi}(x)\times\boldsymbol{\rho} = \mathbf{u}(x) + \boldsymbol{\varphi}\times(y\mathbf{j} + z\mathbf{k})\\ &= U\mathbf{i} + V\mathbf{j} + W\mathbf{k} + (\varphi\mathbf{i} + \psi\mathbf{j} + \chi\mathbf{k})\times(y\mathbf{j} + z\mathbf{k})\\ &= (U + z\psi - y\chi)\mathbf{i} + (V - z\varphi)\mathbf{j} + (W + j\varphi)\mathbf{k}\end{aligned} \quad (1.20)$$

where

$$\begin{aligned}&x = R\sin\theta, \quad y = R\cos\theta,\\ &\mathbf{i} = \mathbf{e}_1, \quad \mathbf{j} = \mathbf{e}_2\cos\theta + \mathbf{m}\sin\theta\\ &\mathbf{k} = -\mathbf{e}_2\sin\theta + \mathbf{m}\cos\theta\end{aligned} \quad (1.21)$$

When expressions (1.21) are substituted into the right-hand side of equality (1.20) and the representations $\mathbf{u} = u\mathbf{e}_1 + v e_2 + w\mathbf{m}$ are substituted into the left-hand side, the equations

$$u = U + R(\psi\cos\theta - \chi\sin\theta), \quad \upsilon = V\cos\theta - W\sin\theta - R\varphi, \quad w = V\sin\theta + W\cos\theta \quad (1.22)$$

can be established which correspond to the use of the Timoshenko shear model, known in the rod theory, for the shell being considered. When Eq. (1.22) are introduced into equalities (1.5), we obtain the relations

$$\begin{aligned}&e_{11} = U' + R(\psi'\cos\theta - \chi'\sin\theta), \quad e_{12} = V'\cos\theta - W'\sin\theta - R\varphi'\\ &e_{21} = -\psi\sin\theta - \chi\cos\theta, \quad e_{22} = 0, \quad \omega_1 = V'\sin\theta + W'\cos\theta, \quad \omega_2 = \varphi\end{aligned} \quad (1.23)$$

from which it follows that the approximation of the displacements by Eq. (1.22) corresponds to the introduction of an assumption on the non-extensibility of the shell in a peripheral direction when it transfers to a perturbed equilibrium state.

Starting from Eq. (1.3) and using relations (1.23) and (1.4), the equation $2\pi RB_{11}U'' = 0$ can be obtained which only has a trivial solution, the individual equation

$$(B_{12} + T^0_{11})R^2\varphi'' - T^0_{22}\varphi = 0 \quad (1.24)$$

describing a torsional FLS from which the bifurcation value $p^*_{(1)}$ (formula (1.11) follows, and also two individual system of equations

$$\begin{aligned}&B_{11}R^2\psi'' - B_{12}W' - (B_{12} + T^0_{22})\psi = 0, \quad B_{12}(W'' + \psi') + 2T^0_{11}W'' = 0\\ &B_{12}R^2\chi'' + B_{12}V' - (B_{12} + T^0_{22})\chi = 0, \quad B_{12}(V'' - \chi') + 2T^0_{11}V'' = 0\end{aligned} \quad (1.25)$$

Both purely shear FLS with the functions χ = const and ψ = const, to which the bifurcation value $p^*_{(3)}$ (formula (1.13) corresponds, as well as bending FLS, which occur in the x,y and x,z planes, are described by these systems of equations, which are absolutely equivalent.

Subsequently considering just the first of the system of Eq. (1.25) obtained, we introduce the displacement function $\mathbf{\Phi}$ into the treatment in the form of the equations

$$\psi = \Phi', \quad W = -\left(1 + \frac{T^0_{22}}{B_{12}}\right)\Phi + \frac{B_{11}R^2}{B_{12}}\Phi'' \quad (1.26)$$

which enable us, in place of the first system of (1.25) when account is taken of relations (1.1), to obtain the resolvent

$$B_{11}R^2\left(1 - \frac{p}{\tilde{B}_{12}}\right)\Phi^{IV} + pR\left(2 - \frac{p}{\tilde{B}_{12}}\right)\Phi'' = 0 \quad (1.27)$$

We next introduce the dimensionless coordinate ζ, the dimensionless transverse shear coefficient k_c and the dimensionless load parameter m using the formulae

$$\zeta = \pi\frac{x}{L}, \quad k_c = \frac{B_1\lambda^2}{B_{12}(1 + \nu_{12})}, \quad \lambda = \frac{\pi R}{L}, \quad pR = \frac{B_1\lambda^2}{1 + \nu_{12}}m \quad (1.28)$$

which enables us to convert Eq. (1.27) to the form

$$(1 - \nu_{12}\nu_{21})(1 - mk_c)\frac{d^4\Phi}{d\zeta^4} + \frac{m}{1 + \nu_{12}}(2 - mk_c)\frac{d^2\Phi}{d\zeta^2} = 0 \quad (1.29)$$

If it is assumed that the end sections of the shell $\zeta = 0$, $\zeta = \pi$ are supported by hinges according to the concepts of the rod theory and the representation $\Phi = \sin\zeta$ is taken for Φ, then the characteristic equation

$$m^2 - m\left[\frac{2}{k_c} + (1 + \nu_{12})(1 - \nu_{12}\nu_{21})\right] + 1 - \nu_{12}\nu_{21} = 0 \quad (1.30)$$

follows from Eq. (1.29).

Table 1

k_c	0.001	0.01	0.1	1.0	10	100	1000
$\nu_{12}=\nu_{21}=0$	$4.99\cdot10^{-4}$	$4.97\cdot10^{-3}$	$4.77\cdot10^{-2}$	0.382	0.600	0.510	0.501
$\nu_{12}=\nu_{21}=0.3$	$4.54\cdot10^{-4}$	$4.52\cdot10^{-3}$	$4.30\cdot10^{-2}$	0.317	0.691	0.601	0.592

The values of the parameter m found from it, which are presented in Table 1, show that there is a very considerable reduction in the critical pressure compared with the value (1.16) if account is taken of the formation of initial stresses T_{11}^0 in the shell due to the action of the pressure on the end sections.

2. Static FLS of a shell under the action of a follower external pressure

We will assume the lateral pressure is a follower pressure and that it is given by the vector

$$X_3\mathbf{m}_* = -p\mathbf{m}_*$$

Moreover, the formula

$$\mathbf{m}_* = E_1\mathbf{e}_1 + E_2\mathbf{e}_2 + E_3\mathbf{m} \tag{2.1}$$

$$E_1 = e_{12}\omega_2 - (1+e_{22})\omega_1, \quad E_2 = e_{21}\omega_1 - (1+e_{11})\omega_2$$
$$E_3 = (1+e_{11})(1+e_{22}) - e_{12}e_{21} \tag{2.2}$$

holds, for small deformations, in the case of the vector of the unit normal to the deformed middle surface.

The variation of the work of the load being considered in the displacement

$$\delta\mathbf{u} = \delta u\mathbf{e}_1 + \delta\upsilon\mathbf{e}_2 + \delta w\mathbf{m}$$

will be equal to

$$\delta R_n = \int_0^L\int_0^{2\pi} X_3\mathbf{m}_*\delta\mathbf{u}Rdxd\theta = -\int_0^L\int_0^{2\pi} p(E_1\delta u + E_2\delta\upsilon + E_3\delta w)Rdxd\theta \tag{2.3}$$

When relations (1.7) and (1.8) are used to describe the neutral equilibrium of the shell instead of Eq. (1.3), it is necessary to set up a variational equation of the following form

$$\int_0^L\int_0^{2\pi}[S_{ij}\delta e_{ij} + S_{i3}\omega_i - p(\omega_1\delta u + \omega_2\delta\upsilon)]Rdxd\theta = 0 \tag{2.4}$$

if it is assumed that the pressure p applied to the end sections remains invariant in direction and, also, when the non-conservative part of the overall end load does not perform work on the possible displacements.

After appropriate reduction, taking account of the equalities

$$\tilde{T}_{11}^0 = -p/2, \quad \tilde{T}_{22}^0 = -p$$

the neutral equilibrium equations

$$f_1 = L_1(u,\upsilon,w) - pu_{,\theta\theta} - pRw_{,x} = 0$$

$$f_2 = L_2(u,\upsilon,w) - \frac{pR^2}{2}\upsilon_{,xx} = 0$$

$$f_3 = L_3(u,\upsilon,w) - \frac{pR^2}{2}w_{,xx} - p(w_{,\theta\theta} - \upsilon_{,\theta}) = 0 \tag{2.5}$$

follow from Eq. (2.4). The first two equations differ from the corresponding equations which have been composed taking account of equalities (1.1) starting from Eq. (1.6).

As in Section 1, we consider three types of solution of the problem of the static FLS of a shell.

1°. Suppose $\partial/\partial\theta = 0$. In this case, Eq. (2.5) take the form

$$f_1 = B_{11}Ru_{,xx} + R(\upsilon_{21}\tilde{B}_{11} - p)w_{,x} = 0$$

$$f_2 = R\left(\tilde{B}_{12} - \frac{p}{2}\right)\upsilon_{,xx} = 0$$

$$f_3 = \upsilon_{12}B_{22}u_{,x} + \tilde{B}_{22}w + \frac{pR^2}{2}w_{,xx} = 0 \tag{2.6}$$

The bifurcation value

$$p^* = 2\tilde{B}_{12} \tag{2.7}$$

follows from the second individual equation of (2.6). This value differs from $p^*_{(1)}$ (formula (1.9)) both with respect to its value and to the nature of the transition to an adjoining state. It is completely analogous to the result obtained[3] for the case of the axial compression of a cylindrical shell and twice as large compared with the value $p^*_{(3)}$ (formula (1.13)), corresponding to the realization of another FLS from the action of an external pressure with a fixed direction on the shell.

The system of equations, consisting of the first and third equations of (2.6), is transformed to the form

$$\frac{pR^3}{2\nu_{21}}w_{,xxx} + \left(\frac{B_1}{\nu_{12}} + pR\right)w_{,x} = 0$$

from which the bifurcation value

$$p^* = \frac{2B_1\nu_{21}}{\nu_{12}R(n^2\lambda^2 - 2\nu_{21})} = \frac{2B_2}{n^2\lambda^2 - 2\nu_{12}}, \quad n = 1, 2, \ldots \tag{2.8}$$

is established. Like $p^*_{(2)}$ (formula (1.11)), this value tends to zero when $n \to \infty$.

2°. In the case when $\partial/\partial x = 0$, Eq. (2.5) take the form

$$f_1 = (\tilde{B}_{12} - p)u_{,\theta\theta} = 0$$

$$f_2 = \tilde{B}_{22}(\upsilon_{,\theta\theta} + w_{,\theta}) = 0, \quad f_3 = \tilde{B}_{22}(\upsilon_{,\theta} + w) + p(w_{,\theta\theta} - \upsilon_{,\theta}) = 0 \tag{2.9}$$

The bifurcation value $p^*_{(3)}$ (formula (1.13)) follows from the first equation of (2.9) which is identical to the first equation of (1.12), and $p^* = 0$ follows from the system of the last two equations of (2.10).

3°. Finally, in order to reveal the beam (rod) FLS of a long shell, starting from the representation of the displacements in the form (1.22), instead of Eq. (1.24) in this case we have the equation

$$(\tilde{B}_{12} - p/2)\varphi'' = 0$$

which yields the bifurcation value (2.7) in the case of purely axial compression by a force per unit length $T_0^{11} = -pR/2$ but, instead of system of equations (1.25), systems of the form

$$B_{11}R^2\psi'' - (B_{12} - pR)(W' + \psi) = 0, \quad B_{12}(W'' + \psi') - pRW'' = 0 \tag{2.10}$$

$$B_{11}R^2\chi'' + (B_{12} - pR)(V' - \chi) = 0, \quad B_{12}(V'' - \chi') - pRV'' = 0 \tag{2.11}$$

by which the bending FLS of a shell are described as for a rod.

If, instead of the functions ψ and W, the resolvent function $\mathbf{\Phi}$ is introduced in accordance with the equations

$$\psi = \Phi', \quad W = -\Phi + \frac{\tilde{B}_{11}R^2}{\tilde{B}_{12} - p}\Phi'' \tag{2.12}$$

then system (2.10) reduces to an ordinary differential equation of the form

$$B_{11}R\Phi^{IV} + p\Phi'' = 0 \tag{2.13}$$

which is identical to the equation of the problem of the stability of a rod in the case of its axial compression within the limits of the classical Bernoulli–Euler model. In the case of the hinged support of the end section of a shell, according to the concepts of the rod theory, it can easily be shown that the boundary conditions

$$\Phi(0) = \Phi(L) = \Phi''(a) = \Phi''(a) = 0 \tag{2.14}$$

are formulated for the function $\mathbf{\Phi}$ when $x = 0$, $x = a$. At the same time, the formula

$$p_4^* = \tilde{B}_{11}\lambda^2 \tag{2.15}$$

which is completely equivalent to the Euler formula for the stability of a rod under conditions of axial compression by the force $p\pi R^2 = -2\pi RT^0_{11}$, follows from Eq. (2.13) for the determining of the bifurcation value of the pressure. This result is fundamentally different from the result in Section 1, which follows from the last formula of (1.28) in the form

$$p^* = \frac{B_1\lambda^2}{R(1+\nu_{12})}m_* \tag{2.16}$$

The value of m_* is determined by the smallest positive root of the quadratic Eq. (1.30), the values of which are shown in Table 1.

Hence, under the action of a follower external pressure on a long cylindrical shell, its loss of stability through a beam bending form is only possible from the axial stress T^0_{11}, which is formed if its end sections are clamped in the same way as the end sections of a conventional rod. If, however, one of its ends is clamped in a fixed manner and the second end is not clamped (a rod fastened in the manner of a cantilever loaded with a "follower" force on the unclamped end) and a "follower" compressive force $P=\pi R^2 p$ is formed in this end section, then, in accordance with the classical results of the theory of stability of rods,[5] a static bending FLS of a shell, which is realized by a transition to a new static equilibrium state is, in general, impossible. To investigate the stability of such a shell, it is necessary to use the dynamic method.[5]

It will be shown below that, when a follower external pressure acts on a shell with a hinged support of the edge, the realization of a dynamic bending FLS, which is fundamentally different from the result in the form of formula (2.15), is possible.

3. Investigation of the perturbed motion of a shell through a beam bending form under the action of an omnidirectional external pressure

In order to derive the equations for the perturbed motion of a shell, loaded in the initial unperturbed state by a follower static lateral pressure, it is necessary to start from the variational equation

$$\int_{\tau_0}^{\tau_1}(\delta\Pi - \delta R_n - \delta K)d\tau = 0 \tag{3.1}$$

where τ is the time and δK is a variation of the kinetic energy, which is equal to (ρ is the density of the shell material and a derivative with respect to τ is denoted by a dot)

$$\delta K = -\rho t\int_0^L\int_0^{2\pi}(\ddot{u}\delta u + \ddot{\upsilon}\delta\upsilon + \ddot{w}\delta w)Rdxd\theta \tag{3.2}$$

Using the results obtained in the preceding sections and starting from relations (3.1) and (3.2), within the limits of the approximation of the displacements u, υ and w by the representations (1.22), we arrive at the perturbed motion equations

$$B_{11}U'' - \rho t\ddot{U} = 0 \tag{3.3}$$

$$\left(B_{12} - \frac{pR}{2}\right)\varphi'' - \rho t\ddot{\varphi} = 0 \tag{3.4}$$

$$B_{11}R^2\psi'' - (B_{12} - pR)(\psi + W') - \rho tR^2\ddot{\psi} = 0$$

$$(B_{12} - pR)W'' + B_{12}\psi' - 2\rho t\ddot{W} = 0 \tag{3.5}$$

$$B_{11}R^2\chi'' + (B_{12} - pR)(V' - \chi) - \rho tR^2\ddot{\chi} = 0$$

$$(B_{12} - pR)V'' - B_{12}\chi' - 2\rho t\ddot{V} = 0 \tag{3.6}$$

for which the boundary conditions when $x=0$, $x=L$ are formulated in the form

$$U' = 0 \text{ when } \delta U \neq 0 \tag{3.7}$$

$$\varphi' = 0 \text{ when } \delta\varphi \neq 0 \tag{3.8}$$

$$\psi' = 0 \text{ when } \delta\psi \neq 0, \quad (B_{12} - pR)W' + B_{12}\psi = 0 \text{ when } \delta W \neq 0 \tag{3.9}$$

$$\chi' = 0 \text{ when } \delta\chi \neq 0, \quad (B_{12} - pR)V' - B_{12}\chi = 0 \text{ when } \delta\chi \neq 0 \tag{3.10}$$

The free longitudinal perturbed vibrations of the shell are described by Eq. (3.3) with boundary conditions (3.7). Within the limits of the geometrically non-linear relations of the theory of momentless shells in the quadratic approximation,[2,7] which are used, it does not contain the parameters of the initial static stress state. When the function $U(x, \tau)$ is represented in the form

$$U(x,\tau) = U_n\sin\frac{n\pi x}{L}e^{-i\omega\tau} \quad \text{on} \quad U(x,\tau) = U_n\cos\frac{n\pi x}{L}e^{-i\omega\tau} \tag{3.11}$$

the formula

$$\omega = \left(\frac{B_{11}n^2\lambda^2}{\rho t}\right)^{1/2}, \quad n = 1, 2, \ldots \tag{3.12}$$

for determining the angular frequency of the free vibrations follows from it.

The analogous torsional vibrations of the shell are described by Eq. (3.4) and, in the case of the boundary conditions $\varphi'(0) = \varphi'(L) = 0$, the formula

$$\omega = n\lambda\left(\frac{B_{12} - pR/2}{\rho t}\right)^{1/2}, \quad n = 1, 2, \dots \tag{3.13}$$

for determining ω follows from it.

These vibrations are stable when $p/2 < \tilde{B}_{12}$, they increase without limit when $p/2 < \tilde{B}_{12}$, and, at the bifurcation value $p^* = 2 < \tilde{B}_{12}$, the shell passes to a perturbed static equilibrium state through a torsional FLS.

The free perturbed flexural vibrations of the shell in the neighbourhood of an initial static stress-strain state in the *x*, *y* and *x*, *z* planes are described by systems of Eq. (3.5) and (3.6), which are absolutely equivalent. The torsional inertial components are taken into account by the last terms of the first equations of these systems. It is well known that they can be neglected without loss of content and accuracy of the phenomena being studied. Therefore, by neglecting the last term in the first equation of system (3.5), for example, which is identical to satisfying it by the introduction of the resolvent function $\mathbf{\Phi}$ in accordance with Eq. (2.12), we reduce the second equation of the system to the form

$$\frac{\partial^4\Phi}{\partial x^4} + \frac{p}{B_{11}R}\frac{\partial^2\Phi}{\partial x^2} - \frac{2\rho t}{B_{12} - pR}\frac{\partial^4\Phi}{\partial x^2\partial\tau^2} + \frac{2\rho t}{B_{11}R}\frac{\partial^2\Phi}{\partial\tau^2} = 0 \tag{3.14}$$

Introducing the new independent variable $\zeta = x/L$ and representing the function $\mathbf{\Phi}$ in the form

$$\Phi = Y(\zeta)e^{i\omega\tau}$$

instead of (3.14), we obtain the ordinary differential equation

$$\frac{d^4Y}{d\zeta^4} + \beta\frac{d^2Y}{d\zeta^2} - \Omega^2 = 0 \tag{3.15}$$

where the dimensionless parameters

$$\beta = L^2\left(\frac{p}{B_{11}R} + \frac{2\rho t\omega^2}{B_{12} - pR}\right), \quad \Omega^2 = \frac{2\rho tL^4\omega^2}{B_{11}R^2} \tag{3.16}$$

have been introduced.

The solution of an equation of the form of (3.16) has been investigated in detail[5] in relation to the problem of the stability of a rod loaded with a “follower” force on the end within the limits of the classical Bernoulli–Euler model. This solution has the form

$$Y(\zeta) = c_1\sin r_+\zeta + c_2\cos r_+\zeta + c_3\operatorname{sh} r_-\zeta + c_4\operatorname{ch} r_-\zeta \tag{3.17}$$

where

$$r_\pm^2 = -\beta/2 \pm \sqrt{\beta^2/4 + \Omega^2} \tag{3.18}$$

In what follows, the investigation of the case of the hinged support of the sections of a shell $\zeta = 0$ and $\zeta = 1$ is of the greatest interest, for which, under the loading conditions considered, a bifurcation value of the pressure (2.15) was established, corresponding to a static transition of the shell to a perturbed equilibrium state through a purely bending mode without the appearance of transverse shear deformations, although the equations used were constructed taking them into account. With the above-mentioned form of clamping, the boundary conditions for the function *Y* will have the form

$$Y(0) = Y''(0) = 0, \quad Y(1) = Y''(1) = 0 \tag{3.19}$$

After subjecting the solutions of (3.17) to the first two conditions of (3.19), we arrive at the following system of two algebraic equations

$$c_2 + c_4 = 0, \quad -r_+^2c_2 + r_-^2c_4 = 0 \tag{3.20}$$

It follows from this system that $c_2 \neq 0$, $c_4 \neq 0$ when the condition $r_+^2 + r_-^2$ is satisfied which, by virtue of relations (3.18) and the second equality of (3.16), takes the form

$$\beta = L^2\left(\frac{p}{B_{11}R} + \frac{2\rho t\omega^2}{B_{12} - pR}\right) = 0$$

whence

$$\omega^2 = \frac{(\tilde{B}_{12} - p)p}{2B_{11}\rho t} \tag{3.21}$$

Consequently, we have a bifurcation value $p^* = \tilde{B}_{12}$, which corresponds to a static loss of stability of the shell through a shear form.

Next, imposing the last two conditions of (3.19) on solution (3.17), we obtain a further two equations

$$c_1\sin r_+ + c_2\cos r_- + c_3\,\mathrm{sh}\,r_- + c_4\,\mathrm{ch}\,r_- = 0$$
$$-c_1 r_+^2\sin r_+ - c_2 r_+^2\cos r_+ + c_3 r_-^2\,\mathrm{sh}\,r_- + c_4 r_-^2\,\mathrm{ch}\,r_- = 0 \quad (3.22)$$

When analysing this system, it is necessary to assume that $r_+^2 + r_-^2 \neq 0$. It then follows from the system of Eq. (3.20) that $c_2 = c_4 = 0$, and the condition for a non-trivial solution of system (3.22) to exist then yields the characteristic equation

$$2(r_+^2 + r_-^2)\sin r_+\,\mathrm{sh}\,r_- = -2\beta\sin r_+\,\mathrm{sh}\,r_- = 0$$

However, according to the assumption which has been made, $2(r_+^2 + r_-^2) = -2\beta \neq 0$). Consequently,

$$\sin r_+\,\mathrm{sh}\,r_- = 0 \quad (3.23)$$

By virtue of the equality $\mathrm{sh}\,ir = i\sin r$, where

$$r^2 = \beta/2 + (\beta^2/4 + \Omega^2)^{1/2} = -r_-^2$$

the resulting Eq. (3.23) has two forms of solution

$$1)\ r_+ = n\pi, \quad n = 0, 1, 2, \ldots; \quad 2)\ r_- = k\pi, \quad k = 0, 1, 2, \ldots \quad (3.24)$$

According to the first solution, using the first formula of (3.19), we obtain the equality

$$\beta = -n^2\pi^2 + \frac{\Omega^2}{n^2\pi^2} \quad (3.25)$$

from which, when expressions (3.16) are substituted, we arrive at the formula

$$\omega^2 = -\frac{pL^2 + B_{11}Rn^2\pi^2}{2\rho t L^2 B_{11}R}\left(\frac{1}{B_{12} - pR} - \frac{L^2}{B_{11}R^2n^2\pi^2}\right)^{-1}$$

It follows from this formula that the perturbed vibrations of the shell involving a bending mode become unstable, that is, $\omega^2 = \infty$ when the equality

$$B_{11}R^2n^2\pi^2 = L^2(B_{12} - pR)$$

is satisfied.

It follows from this that

$$p_{(1)}^{*\partial} = \tilde{B}_{12} - \tilde{B}_{11}\lambda^2 n^2 = p_c^* - n^2 p_u^*, \quad n = 0, 1, 2, \ldots \quad (3.26)$$

for the critical pressure in the case of dynamic loss of stability through a bending form. The values of p_c^* and p_u^* are determined using formulae (1.13) and (2.15).

By analogy with equality (3.23), starting out from the second equation of (3.24), we arrive at the formula

$$p_{(2)}^{*\partial} = \tilde{B}_{12} + k^2\tilde{B}_{11}\lambda^2, \quad \text{on} \quad p_{(2)}^* = p_c^* + k^2 p_u^*, \quad k = 0, 1, 2, \ldots \quad (3.27)$$

Since, in the case of practical structures $B_{12} \gg B_{11}\lambda^2$, we have $p_{(2)}^{*\partial} > p_{(1)}^{*\partial}$, and the minimum positive value $p^{*\partial} > p_{(1)}^{*\partial}$, has to be determined using formula (3.26) for the critical value n_* which corresponds to the minimum of $p_{(1)}^{*\partial}$.

Hence, when a follower omnidirectional hydrostatic pressure acts on a long cylindrical shell, which is closed in the end sections and which are supported by hinges according to the concepts of the rod theory (Fig. 1), three forms of loss of stability can occur: (1) a static loss of stability through a bending form from the action of a compressive axial force $P = \pi R^2 p$, since, under the conditions considered, its non-conservative part cannot perform work on the displacements V and W, (2) loss of stability which is also static but occurs through a

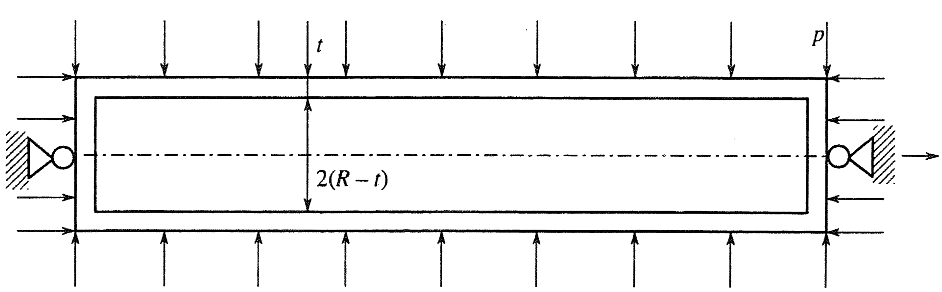

Fig. 1.

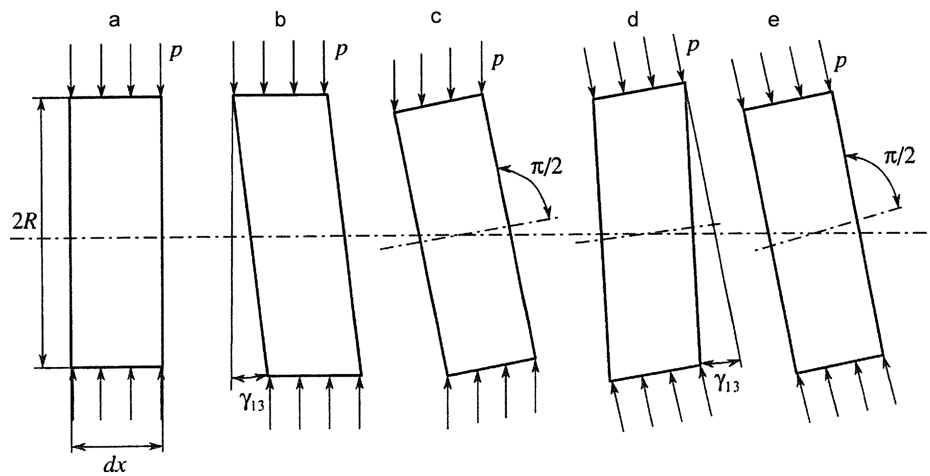

Fig. 2.

purely shear mode, and the corresponding critical load is independent of the length of the shell, and (3) dynamic loss of stability which is occurs through a bending-shear mode and can only be revealed by the dynamic method.[5]

The positions of an element of a shell of length *dx* are shown in Fig. 2 for the initial undeformed state (*a*) and for the perturbed state after loss of stability: (*b*) through a purely shear form, the realization of which is possible under the action of an external pressure *p* both of a fixed direction as well as of a pressure which remains normal to the surface of the shell, (*c*) through a purely bending form under the action of a pressure of fixed direction,[2] (*d*) for a dynamic loss of stability which is possible in the case of the action of a follower pressure and only when transverse displacements are taken into account using the Timoshenko kinematic rod model (or any other improved model). Loss of stability through such a form is impossible if the transverse displacements in the cross sections of a rod are not taken into account and the perturbed bent state is described by the classical Bernoulli–Euler model (*f*).

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